Math 280 Multivariable Calculus Fall 2012

Lagrange Multiplier Example

Here is a carefully presented solution to Exercise #8 in Section 12.8.

Problem: Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

Translation: First we rephrase the problem in a form more amenable to mathematical analysis.

- Use symbols to name the appropriate quantities. Since we are interested in distances from points to the origin we designate an arbitrary point in the plane by $P(x, y)$ and the origin by $O(0, 0)$.
- Convert into mathematics. Using the distance formula we see that the distance between P and O is $\sqrt{(x-0)^2 + (y-0)^2}$.
- Think ahead to simplify later work. Since the function $g(t) = \sqrt{t}$ is strictly increasing, then we know that a maximum output of g occurs when the input is as large as possible and that a minimum output occurs when the input is as small as possible. Thus, any point $P(x, y)$ that maximizes (or minimizes) $x^2 + y^2$ is also a point that maximizes (or minmizes) $\sqrt{x^2 + y^2}$. Since derivatives of the former do not require the chain rule, it is simpler to convert our problem to the following constrained optimization.

Problem Statement:
Maximize (and Minimize) $f(x, y) = x^2 + y^2$
Subject to the constraint: $g(x, y) = x^2 + xy + y^2 = 1$.
Solution:

- Using the method of Lagrange Multipliers, we know that any maximizers and minimizers for this constrained optimization problem are points (x, y) that satisfy the constraint equation and for which there is a scalar λ satisfying $\nabla f(x, y) = \lambda \nabla g(x, y)$.Thus we wish to find all points (x, y) and all λ that satisfy the following three equations

  1. $f_x(x, y) = \lambda g_x(x, y)$
  2. $f_y(x, y) = \lambda g_y(x, y)$
  3. $x^2 + xy + y^2 = 1$

- For this problem our three equations are:

$$\begin{aligned} 2x &= \lambda(2x + y) \\ 2y &= \lambda(x + 2y) \\ 1 &= x^2 + xy + y^2 \end{aligned}$$

- We will use the symmetry that is evident in the first two equations (the second is the first with the roles of x and y reversed) by solving for λ in both.
- First note that if $2x+y = 0$, then the first equation implies that $x = 0$ and so $2x+y = 2(0)+y = 0$ so $y = 0$ as well. Checking the input $(x, y) = (0, 0)$ in the third equation shows us that the point $(0, 0)$ is not in the constrained domain of our function f. Thus, for our problem, we may assume $2x + y \neq 0$ (which allows us to divide by it when we solve for λ).

- In a similar fashion,but using the second equation, we may also assume that $x + 2y \neq 0$.
- This allows us to rewrite our three equations as

$$\begin{aligned} \frac{2x}{2x+y} &= \lambda \\ \frac{2y}{x+2y} &= \lambda \\ 1 &= x^2 + +xy + y^2 \end{aligned}$$

- Setting the two λ's in the top two equations allows us to deduce that any point (x, y) solving all three equations must also satisfy the equation.

$$\frac{2x}{2x+y} = \frac{2y}{x+2y}.$$

  (Technically, this is the same as solving for λ in one equation and substituting the result into the other equation.)
- Doing a bit of algebra with this we obtain

$$\begin{aligned} 2x\,(x+2y) &= 2y\,(2x+y) \\ 2x^2 + 4xy &= 4xy + 2y^2 \\ 2x^2 &= 2y^2 \\ x^2 &= y^2 \\ x &= y \text{ or } x = -y \end{aligned}$$

- We now know that any point (x, y) satisfying all three equations falls into one of the two cases: $x = y$ or $\quad x = -y$

  1. If $x = y$, the the third equation tells us that

$$\begin{aligned} 1 &= x^2 + x\,(x) + (x)^2 \\ 1 &= 3x^2 \\ x &= \frac{1}{\sqrt{3}} \text{ or } x = -\frac{1}{\sqrt{3}} \end{aligned}$$

     Since $x = y$ in this case, we get the two points $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ as possible maximizers or minimizers.
  2. If $x = -y$, the the third equation tells us that

$$\begin{aligned} 1 &= x^2 + x\,(-x) + (-x)^2 \\ 1 &= x^2 \\ x &= 1 \text{ or } x = -1 \end{aligned}$$

     Since $x = -y$ in this case, we get the two points $(1, -1)$ and $(-1, 1)$.

- Thus a complete list of possible maximizers and minimizers for our constrained optimization problem is: $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), (1, -1)$ and $(-1, 1)$.

- Evaluating f at each of these points gives

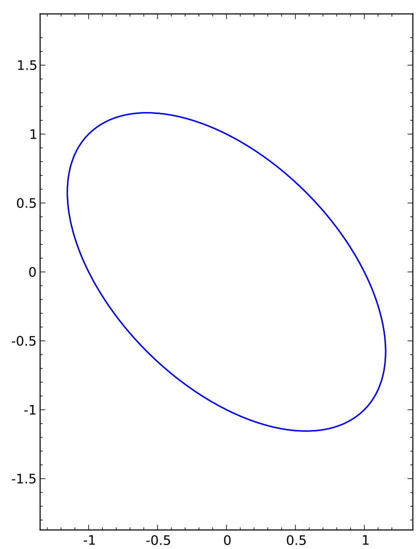
  1. $f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$
  2. $f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$
  3. $f(1, -1) = 1 + 1 = 2$
  4. $f(-1, 1) = 1 + 1 = 2$

- So far we have $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ are both minimizers of $f(x, y) = x^2 + y^2$ with common minimum value of $\frac{2}{3}$ and

  $(1, -1)$ and $(-1, 1)$ are both maximizers of $f(x, y) = x^2 + y^2$ with common maximum value of 2.

- Going back to the beginning we now see that the points on the curve that are nearest to the origin are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and the points that are farthest from the origin are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. The respective smallest and largest distances are $\sqrt{\frac{2}{3}}$ and $\sqrt{2}$.

Using technology we can plot the graph of the constraint equation and see that it is an ellipse that has been rotated about the origin.



Notice where the four points are in this plot.